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Scattering Theory, Cross Sections, and the Lattice of Propositions

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The asymptotic conditions for the nonrelativistic quantum scattering of a particle by a center of force are derived in terms of a metric on the space of states on a complete orthocomplemented lattice. The flux of particles scattered into a cone $\mathscr{C}$ per unit incident flux, averaged over all displacements of the center of force at right angles to the axis of the incident beam, is expressed in terms of the differential cross section $d\sigma/d\omega$ when the motion is classical, and in terms of the scattering amplitude f when the motion is quantum mechanical. This enables the usual identification $d\sigma/d\omega = |f|^2$ to be made.

1. INTRODUCTION

The motivation for this work is a satisfactory derivation of the relation between the differential cross section and the scattering amplitude in the quantum theory of scattering. In the case of the scattering of a particle by a fixed center of force which conserves energy this is the well-known expression

$$\frac{d\sigma(\theta,\varphi)}{d\omega} = |f(\theta,\varphi)|^2 \tag{1}$$

where $d\sigma(\theta,\varphi)/d\omega$ is the differential cross section and $f(\theta,\varphi)$ the scattering amplitude.

It has long been realized that the time-independent derivation of (1) found in most introductory textbooks on quantum scattering theory is inadequate. For in actual scattering experiments the beams are of finite extent, and the detectors are, of course, placed outside the beam. Further, modern experiments using such techniques as time-of-flight devices clearly

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show the time-dependent nature of the scattering process, as do experiments using pulsed beams. For such reasons wave packet approaches to the quantum theory of scattering were developed, for example, in the book by Goldberger and Watson (1964).

Time-dependent derivations of (1) usually assume that a particle is described by a wave packet, or equivalently, a ket vector. The wave packet may be assumed to have a cross-sectional area imposed by the source (Farina 1973), or the beam may be modeled by a stream of particles each associated with its own wave packet (Taylor, 1972; Amrein, Jauch, and Sinha, 1977). All these derivations assume that each particle is in a pure state. However, there is nothing in modern formulations of quantum mechanics which requires that systems are in pure states, and the experimental conditions do not give us sufficient knowledge of the particles in the incident beam or pulse to enable us to make such an assertion.

It seems to the author that a satisfactory derivation of (1) must go back to the first principles of quantum mechanics. An increasingly accepted version of these principles is the "lattice of propositions," or "quantum logic" (see, for example, Jauch, 1968; Piron, 1976; or Mackey, 1963). A general system of axioms has been proposed by Piron (1976), but in the case of the nonrelativistic quantum mechanics of a system consisting of a fixed, finite number of mutually interacting particles it is sufficient to assume that the lattice of propositions corresponds to the lattice of closed subspaces of a Hilbert space. Gleason's famous theorem then enables us to connect the probabilities for any state with probabilities for pure states.

The plan of the paper is as follows. In Section 2 we discuss states on a complete, orthocomplemented lattice $\mathscr{L}$; that is, probability measures on such a lattice. The results derived in this section apply equally to classical and quantum mechanics. In it we describe a metric on the convex set of states, so that the states form a metric space $M(\mathscr{L})$. In Section 3 we derive some further results in the special case when $\mathscr{L}$ is the lattice $\mathscr{L}(\mathscr{H})$ of closed subspaces of a Hilbert space $\mathscr{H}$, appropriate for nonrelativistic quantum mechanics. Some of these results are not used in the subsequent derivation of (1), but it is hoped they will be of interest in their own right. For example, it is shown that any state may be approximated by a finite convex linear combination of pure states defined by vectors belonging to a dense subset of $\mathscr{H}$.

From Section 4 onwards we deal with the special case of the scattering of a particle by a fixed center of force. In Section 4 itself some standard results of quantum scattering theory are extended to states other than pure states, position and momentum are discussed in Section 5, scattering into cones in Section 6, the translation of states in space in Section 7, and the momentum distribution in the initial state in Section 8. Section 9 contains the derivation of the relation (77) which connects the flux $\sigma(\mathscr{C})$ into a cone $\mathscr{C}$ with the probability of the final momentum being in $\mathscr{C}$. A classical calculation leading to the expression (82) of $\sigma(\mathscr{C})$ in terms of the differential cross section $d\sigma/d\omega$ is performed in Section 10. The same expression is shown to be true in Section 11 when the motion is quantum mechanical, provided the identification (1) is made, and our conclusions discussed in Section 12.

2. STATES ON $\mathscr{L}$

A state μ on a complete orthocomplemented lattice $\mathscr{L}$ of propositions for a given physical system (classical or quantum mechanical) is assumed to be a probability measure on $\mathscr{L}$. Thus μ assigns to each proposition p in $\mathscr{L}$ a probability $P[p|\mu]$. The set $M = M(\mathscr{L})$ of all such states forms a convex set; that is, if $\{\mu_i\}_{i\in I}$ (I a countable index set) is a set of such states and $\{\lambda_i\}_{i\in I}$ is a set of positive numbers whose sum is unity then the expression

$$P[p|\mu] = \sum_{i\in I} \lambda_i P[p|\mu_i] \qquad (p \in \mathscr{L}) \tag{2}$$

defines a state μ. (2) can be written more briefly as

$$\mu = \sum_{i\in I} \lambda_i \mu_i \tag{3}$$

If μ_1 and μ_2 are any two states the quantity

$$d(\mu_1, \mu_2) = \sup_{p\in\mathscr{L}} |P[p|\mu_1] - P[p|\mu_2]| \tag{4}$$

is a well-defined number in the interval [0,1]. It is easy to verify that (4) defines a metric on M, so that M is a metric space. In fact such a metric on a classical probability space is known to statisticians as the "total variation distance" (see, for example, Huber, 1981, p. 34).

Proposition 2.1. Suppose (3) is valid and $\{\mu_i'\}_{i\in I}$ is a second set of states in 1:1 correspondence with the states $\{\mu_i\}_{i\in I}$. If a state μ' is defined by

$$\mu' = \sum_{i\in I} \lambda_i \mu_i' \tag{5}$$

then

$$d(\mu, \mu') \leqslant \sum_{i\in I} \lambda_i d(\mu_i, \mu_i') \tag{6}$$

Proof. If $p \in \mathscr{L}$,

$$|P[p|\mu]-P[p|\mu']| \leqslant \sum_{i \in I} \lambda_i |P[p|\mu_i]-P[p|\mu_i']|$$
$$\leqslant \sum_{i \in I} \lambda_i d(\mu_i, \mu_i')$$

Since the right-hand side is independent of p, the result follows from the definition (4). ■

Corollary(2.1a). If μ and μ' are as in (3) and (5), and if $d(\mu_i, \mu_i') \leqslant \varepsilon(\varepsilon > 0)$ for all $i \in I$ then $d(\mu, \mu') \leqslant \varepsilon$.

Proof. This follows from (6) since $\sum_{i \in I} \lambda_i = 1$. ■

A *symmetry*, or *automorphism*, of $\mathscr{L}$ is a bijection W of $\mathscr{L}$ to $\mathscr{L}$ which preserves the lattice structure. It has an inverse mapping W^{-1} which is also a symmetry. Thus $W\mathscr{L} = \mathscr{L} = W^{-1}\mathscr{L}$.

For any μ in M we can define a mapping $W\mu$ of $\mathscr{L}$ into $\mathbb{R}$ by

$$P[p|W\mu] = P[W^{-1}p|\mu] \ \forall p \in \mathscr{L} \tag{7}$$

It is well known, and easy to verify, that $W\mu$ is a state.

Proposition 2.2. The mapping of M into itself induced by a symmetry W preserves the metric. That is, if μ_1 and μ_2 are states then

$$d(W\mu_1, W\mu_2) = d(\mu_1, \mu_2) \tag{8}$$

Proof.

$$d(W\mu_1, W\mu_2) = \sup_{p \in \mathscr{L}} |P[p|W\mu_1] - P[p|W\mu_2]|$$
$$= \sup_{p \in \mathscr{L}} |P[W^{-1}p|\mu_1] - P[W^{-1}p|\mu_2]|$$
$$= \sup_{p \in W^{-1}\mathscr{L}} |P[p|\mu_1] - P[p|\mu_2]|$$
$$= \sup_{p \in \mathscr{L}} |P[p|\mu_1] - P[p|\mu_2]|$$
$$= d(\mu_1, \mu_2)$$ ■

A sequence $\{\mu_n\}_{n=1}^{\infty}$ of states in M *converges* to a state μ in M if $d(\mu, \mu_n) \to 0$ as $n \to \infty$, this being the usual definition of convergence in a metric space. We can denote the limit μ by $\lim_{n\to\infty}\mu_n$. The following proposition shows that we can interchange limit and sum for infinite convex linear combinations.

Proposition 2.3. Let μ be given by (3). Suppose further that for each $i \in I$ there is a sequence $\{\mu_{i,n}\}_{n=1}^{\infty} \subseteq M$ such that

$$\mu_i = \lim_{n\to\infty} \mu_{i,n} \tag{9}$$

Then if a sequence of states $\{\mu_n\}_{n=1}^{\infty}$ is defined by

$$\mu_n = \sum_{i\in I} \lambda_i \mu_{i,n} \tag{10}$$

μ_n tends to μ as $n \to \infty$. That is,

$$\lim_{n\to\infty} \sum_{i\in I} \lambda_i \mu_{i,n} = \sum_{i\in I} \lambda_i \lim_{n\to\infty} \mu_{i,n} \tag{11}$$

Proof. From (6) (Proposition 2.1)

$$d(\mu, \mu_n) \leqslant \sum_{i\in I} \lambda_i d(\mu_i, \mu_{i,n}) \tag{12}$$

The right-hand side of (12) is either a finite sum or a uniformly and absolutely convergent series by comparison with $\sum_{i\in I}\lambda_i$. In either case we can let $n \to \infty$ term by term on the right-hand side of (12), when $d(\mu, \mu_n) \to 0$. ■

If I is infinite and we terminate the sum in (3) we do not get a state. We can, however, approximate μ by finite convex linear combinations of the μ_i as the next proposition shows.

Proposition 2.4. Let μ be given by (3) with $I = \mathbb{N}$, the positive integers, and let n be a positive integer. For each $i \in \mathbb{N}$ define $\lambda_{i,n}$ by

$$\lambda_{i,n} = \left(\sum_{j=1}^{n} \lambda_j \right)^{-1} \lambda_i \tag{13}$$

Then a state μ_n is defined by

$$\mu_n = \sum_{i=1}^{n} \lambda_{i,n} \mu_i \tag{14}$$

Moreover, $d(\mu, \mu_n) \to 0$ as $n \to \infty$.

Proof. From (13) $\lambda_{i,n} > 0$ and $\Sigma_{i=1}^{n}\lambda_{i,n} = 1$, hence (14) defines a state. Further, for all $p \in \mathscr{L}$, (2) and (14) show that

$$P[p|\mu] - P[p|\mu_n] = \sum_{i=1}^{n} (\lambda_i - \lambda_{i,n}) P[p|\mu_i] + \sum_{i=n+1}^{\infty} \lambda_i P[p|\mu_i]$$

Since (13) implies that $\lambda_i < \lambda_{i,n}$ and $0 \leqslant P[p|\mu_i] \leqslant 1$ we find that

$$\begin{aligned} |P[p|\mu] - P[p|\mu_n]| &\leqslant \sum_{i=1}^{n} (\lambda_{i,n} - \lambda_i) + \sum_{i=n+1}^{\infty} \lambda_i \\ &= 1 - \sum_{i=1}^{n} \lambda_i + \sum_{i=n+1}^{\infty} \lambda_i \\ &= 2 \sum_{i=n+1}^{\infty} \lambda_i \end{aligned}$$

hence

$$d(\mu, \mu_n) \leqslant 2 \sum_{i=n+1}^{\infty} \lambda_i$$

and so $d(\mu, \mu_n) \to 0$ as $n \to \infty$. ■

Proposition 2.5. Suppose $\{A_t\}_{t\in\mathbb{R}}$ and $\{B_t\}_{t\in\mathbb{R}}$ are two families of mappings of M into itself, and that further B_t is isometric for all t in $\mathbb{R}$ (that is, it preserves the metric on M). If μ, μ_1 and μ_2 are states such that

$$d(A_t\mu, B_t\mu_1) \to 0 \qquad \text{and} \qquad d(A_t\mu, B_t\mu_2) \to 0$$

when $t \to -\infty$, then $\mu_1 = \mu_2$. A similar result holds when $t \to +\infty$.

Proof.

$$\begin{aligned} d(\mu_1, \mu_2) &= d(B_t\mu_1, B_t\mu_2) \\ &\leqslant d(B_t\mu_1, A_t\mu) + d(A_t\mu, B_t\mu_2) \\ &\to 0 \; as \; t \to -\infty \end{aligned}$$

Hence $d(\mu_1, \mu_2) = 0$ and so $\mu_1 = \mu_2$. ■

Continuous Distributions. Suppose $\mathscr{P}$ is a family of propositions of the form "$\mathbf{k} \in \mathscr{R}$" where $\mathscr{R}$ ranges over all Borel subsets of $\mathbb{R}^N$. If for μ in M

there is a function $P[\cdot|\mu]$ such that, for arbitrary $\mathscr{R}$ in $\mathbb{R}^N$

$$P[\mathbf{k} \in \mathbb{R}|\mu] = \int_{\mathscr{R}} P[\mathbf{k}|\mu]\, d^N\mathbf{k} \tag{15}$$

then $\mathbf{k}$ has a continuous distribution $P[\cdot|\mu]$ in the state μ. The function $P[\cdot|\mu]$ may depend on $\mathscr{P}$ as well as μ.

Total Sets. We shall call a subset M_0 of M *total* in M if the set of finite convex linear combinations of states in M_0 is dense in M.

Proposition 2.6. Suppose μ is given by (3) and $\mathbf{k}$ has a continuous distribution $P[\cdot|\mu_i]$ in the state μ_i for each i in I. Then $\mathbf{k}$ has a continuous distribution in the state μ and moveover

$$P[\cdot|\mu] = \sum_{i \in I} \lambda_i P[\cdot|\mu_i] \tag{16}$$

Proof. If I is finite $P[\cdot|\mu]$ can be defined by (16) and the result follows, so suppose I is infinite. Then (3) becomes

$$\mu = \sum_{i=1}^{\infty} \lambda_i \mu_i \tag{17}$$

Since $\mathbf{k}$ has a continuous distribution in each state μ_i we can define a sequence $\{P_n[\cdot|\mu]\}_{n=1}^{\infty}$ of functions on $\mathscr{L}_1(\mathbb{R}^N)$ by

$$P_n[\cdot|\mu] = \sum_{i=1}^{n} \lambda_i P[\cdot|\mu_i] \tag{18}$$

It is easy to see that $\{P_n[\cdot|\mu]\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathscr{L}_1(\mathbb{R}^N)$ and so converges to a limit in $\mathscr{L}_1(\mathbb{R}^N)$ which we denote by $P[\cdot|\mu]$. Moreover, $P_n[\cdot|\mu]$ converges almost everywhere to $P[\cdot|\mu]$, and therefore

$$P[\cdot|\mu] = \sum_{i=1}^{\infty} \lambda_i P[\cdot|\mu_i] \tag{19}$$

It follows from (17) that

$$\begin{aligned} P[\mathbf{k} \in \mathscr{R}|\mu] &= \sum_{i=1}^{\infty} P[\mathbf{k} \in \mathscr{R}|\mu_i] \\ &= \sum_{i=1}^{\infty} \lambda_i \int_{\mathscr{R}} P[\mathbf{k}|\mu_i]\, d^N\mathbf{k} \end{aligned} \tag{20}$$

Since $P_n[\cdot|\mu]$ is bounded by $P[\cdot|\mu]$ the dominated convergence theorem shows that

$$\int_{\mathscr{R}} P[\mathbf{k}|\mu]\,d^N\mathbf{k} = \lim_{n\to\infty} \int_{\mathscr{R}} P_n[\mathbf{k}|\mu]\,d^N\mathbf{k} = \sum_{i=1}^{\infty} \int_{\mathscr{R}} \lambda_i P[\mathbf{k}|\mu_i]\,d^N\mathbf{k} \tag{21}$$

by (18). The right-hand sides of (20) and (21) are equal, hence so are the left-hand sides. Thus **k** has a continuous distribution $P[\cdot|\mu]$ in the state μ, and (16) follows from (19). ■

3. STATES ON $\mathscr{L}(\mathscr{H})$

In nonrelativistic quantum mechanics the lattice of propositions corresponds to the lattice $\mathscr{L}(\mathscr{H})$ of subspaces of a Hilbert space $\mathscr{H}$. That is to say, there is an isomorphism between propositions and $\mathscr{L}(\mathscr{H})$ according to

$$p \leftrightarrow \mathscr{U}_p \leftrightarrow E_p \tag{22}$$

in (22) $\mathscr{U}_p$ is the subspace corresponding to p and E_p is the projection operator onto $\mathscr{U}_p$. Clearly $\mathscr{U}_p = E_p\mathscr{H}$.

Each unit vector f in $\mathscr{H}$ defines a pure state $\mu(f)$ according to

$$P[p|\mu(f)] = \|E_p f\|^2 \tag{23}$$

(23) will be written in the briefer form

$$P[p|f] = \|E_p f\|^2 \tag{24}$$

Proposition 3.1. The distance $d(\mu(f), \mu(g))$ between two pure states defined by the unit vectors f and g satisfies the inequality

$$d(\mu(f), \mu(g)) \leqslant 2\|f - g\| \tag{25}$$

Proof. For all p in $\mathscr{L}(\mathscr{H})$,

$$\begin{aligned} |P[p|f] - P[p|g]| &= |\|E_p f\|^2 - \|E_p g\|^2| \\ &= (\|E_p f\| + \|E_p g\|)\,|\|E_p f\| - \|E_p g\|| \\ &\leqslant 2\|E_p f - E_p g\| \\ &= 2\|E_p(f-g)\| \\ &\leqslant 2\|f-g\| \end{aligned}$$

whence (25) follows from (4). ■

Gleason's Theorem. This well-known theorem states that every μ in M can be expressed in the form (3) where, for each i in the countable index set I, μ_i is pure. That is, given μ in M there is a countable set $\{f_i\}_{i\in I}$ of unit vectors and a corresponding set $\{\lambda_i\}_{i\in I}$ of positive numbers, whose sum is unity, such that

$$\mu = \sum_{i\in I} \lambda_i \mu(f_i) \tag{26}$$

Equivalently,

$$P[p|\mu] = \sum_{i\in I} \lambda_i P[p|f_i] \quad \forall p \in \mathscr{L} \tag{27}$$

where $\mathscr{L} = \mathscr{L}(\mathscr{H})$. Thus by Proposition 2.4 the pure states are total in M.

Proposition 3.2. Suppose that μ is given by (26) while μ' in M is given by

$$\mu' = \sum_{i\in I} \lambda_i \mu(f_i') \tag{28}$$

where $\{f_i'\}_{i\in I}$ is also a set of unit vectors. Then

$$d(\mu, \mu') \leqslant 2 \sum_{i\in I} \lambda_i \|f_i - f_i'\| \tag{29}$$

Proof. By (6) (Proposition 2.1), and then (25),

$$d(\mu, \mu') \leqslant \sum_{i\in I} \lambda_i d\big(\mu(f_i), \mu(f_i')\big) \leqslant 2 \sum_{i\in I} \lambda_i \|f_i - f_i'\| \qquad \blacksquare$$

Proposition 3.3. Let $\mathscr{U}$ be a linear manifold dense in $\mathscr{H}$. Then the set of pure states defined by unit vectors in $\mathscr{U}$ is total in M.

Proof. Suppose μ is any state; then it can be expressed in the form (26). Since $\mathscr{U}$ is dense in $\mathscr{H}$ it is easy to show that, given $\varepsilon > 0$, corresponding to each unit vector f_i there is a unit vector f_i' in $\mathscr{U}$ such that $\|f_i - f_i'\| \leqslant \frac{1}{2}\varepsilon$. Hence defining μ' by (28) the inequality (29) yields $d(\mu, \mu') \leqslant \varepsilon$. Taking account of Proposition 2.4 if I is infinite we see that finite convex linear combinations of states of the form $\mu(f_i)$ with f_i in $\mathscr{U}$ form a dense set in M; that is, such states are total in M. ■

A simple consequence of Proposition 3.3 is, for example, that any state is experimentally indistinguishable from a state which is a finite convex linear combination of pure states defined by Schwarz functions if there is an $\varepsilon > 0$ such that $P[p|\mu]$ is not measurable beyond an absolute accuracy of ε for every p in $\mathscr{L}$.

Induced States. Suppose W is an isometric operator on $\mathscr{H}$. Then W induces a mapping of M into itself if we define, for μ in M, $W\mu$ by

$$W\mu = \sum_{i \in I} \lambda_i \mu(Wf_i) \tag{30}$$

where μ is given by (26). For $\|Wf_i\| = \|f_i\| = 1$ for each i in I and $\sum_{i \in I} \lambda_i = 1$, hence (30) defines a state.

A special case occurs if W is unitary. Then W induces an automorphism of $\mathscr{L}(\mathscr{H})$ into itself according to

$$E_p\mathscr{H} \leftrightarrow WE_p\mathscr{H}, \qquad E_p \leftrightarrow WE_pW^* \tag{31}$$

The inverse mapping is

$$E_p\mathscr{H} \leftrightarrow W^*E_p\mathscr{H}, \qquad E_p \leftrightarrow W^*E_pW \tag{32}$$

If we denote the proposition corresponding to $WE_p\mathscr{H}$ by Wp, W^*E_p corresponds to $W^{-1}p$, and for all p in $\mathscr{L}$,

$$\begin{aligned} P[p|W\mu] &= \sum_{i \in I} \lambda_i \|E_pWf_i\|^2 \\ &= \sum_{i \in I} \lambda_i \|W^*E_pWf_i\|^2 \\ &= P\left[W^{-1}p|\mu\right] \end{aligned}$$

which is (7). Hence by Proposition 2.2 the mapping $\mu \to W\mu$ is isometric.

Proposition 3.4. If $\{A_n\}_{n=1}^{\infty}$ is a sequence of isometric operators on $\mathscr{H}$ which converges strongly to an isometric operator A on $\mathscr{H}$ then the sequence $\{A_n\mu\}_{n=1}^{\infty}$ converges to $A\mu$.

Proof. From (29), with f_i and f_i' replaced by Af_i and A_nf_i, respectively,

$$d(A\mu, A_n\mu) \leqslant 2 \sum_{i \in I} \lambda_i \|Af_i - A_nf_i\| \tag{33}$$

Now when $n \to \infty$ $A_nf_i \to Af_i$ for each i in I, and $\|Af_i - A_nf_i\| \leqslant \|Af_i\| + \|A_nf_i\| \leqslant 2$, so if the right-hand side of (31) is not finite then it is uniformly convergent. We can therefore let $n \to \infty$ term by term, so $d(A\mu, A_n\mu) \to 0$ as $n \to \infty$. ■

Proposition 3.5. If $\{A_t\}_{t \in \mathbf{R}}$ is a set of isometric operators on $\mathscr{H}$ and A is another isometric operator on $\mathscr{H}$ such that

$$\operatorname*{s-lim}_{t \to \pm\infty} A_t = A$$

then, when $t \to \pm\infty$, $d(A\mu, A_t\mu) \to 0$ for any μ in M.

Proof. As for Proposition 3.4.

Proposition 3.6. Suppose $\{A_t\}_{t \to \mathbf{R}}$ and $\{B_t\}_{t \in \mathbf{R}}$ are two families of isometric operators on $\mathscr{H}$. If, for all f in $\mathscr{H}$,

$$\lim_{t \to -\infty} \|A_t f - B_t f\| = 0$$

then

$$\lim_{t \to -\infty} d(A_t\mu, B_t\mu) = 0$$

for any μ in M.

The same is true when $t \to +\infty$.

Proof. By (29) with f_i and f_i' replaced by $A_t f_i$ and $B_t f_i$, respectively,

$$d(A_t\mu, B_t\mu) \leqslant 2 \sum_{i \in I} \lambda_i \|A_t f_i - B_t f_i\|$$

The proof now proceeds as for that of Proposition 3.4. ■

4. THE ASYMPTOTIC CONDITION

From now on we shall assume that our system consists of a single particle moving under a central potential. The Hilbert space $\mathscr{H}$ is now $\mathscr{L}_2(\mathbb{R}^3)$, and we denote the evolution operators for the free and perturbed motion by U_t and V_t, respectively, where t is the time. We assume that the potential is sufficiently well behaved for all the results of elementary scattering theory to be valid, such as the existence of the wave and scattering operators, and asymptotic completeness (Amrein, Jauch, and Sinha, 1977). The results of Section 3 enable us to easily describe the asymptotic condition for potential scattering, which we do in this section.

The initial (freely moving) state at time t has the form (26) where each f_i evolves under U_t. If μ is the state which would exist at $t = 0$ if there were no interaction the initial state at time t is $U_t\mu$, and given by

$$U_t\mu = \sum_{i \in I} \lambda_i \mu(U_t f_i) \tag{34}$$

The wave operators $\Omega \pm$ on $\mathscr{H}$ are defined by

$$\Omega_{\pm} = \operatorname*{s-lim}_{t \to \pm\infty} V_t^* U_t \tag{35}$$

and as stated above we assume that these limits exist. Since $\Omega_{\pm}$ are isometric operators on $\mathscr{H}$ we can define states $\Omega_{\pm}\mu$ [cf. (30) with $W = \Omega_{\pm}$] which, under the full evolution operator V_t, evolve into the states $V_t\Omega_{\pm}\mu$ given by

$$V_t\Omega_{\pm}\mu = \sum_{i \in I} \lambda_i \mu(V_t\Omega_{\pm} f_i) \tag{36}$$

Proposition 4.1. For any μ in M,

$$\lim_{t \to \pm\infty} d(U_t\mu, V_t\Omega_{\pm}\mu) = 0 \tag{37}$$

Proof. From (35), if $f \in \mathscr{H}$,

$$\lim_{t \to \pm\infty} \|U_t f - V_t\Omega_{\pm} f\| = 0$$

Hence the result follows by Proposition 3.6 with $A_t = U_t$, $B_t = V_t\Omega_{\pm}$, both of which are isometric. ∎

The following proposition shows that $\Omega_{-}\mu$ is the only state having the asymptotic property expressed by (37) with the lower sign. That is, $V_t\Omega_{-}\mu$ is the one and only state of full motion which is asymptotic to $U_t\mu$ as $t \to -\infty$.

Proposition 4.2. Suppose μ_1 and μ_2 are states such that

$$d(U_t\mu, V_t\mu_1) \to 0 \quad \text{and} \quad d(U_t\mu, V_t\mu_2) \to 0$$

as $t \to -\infty$. Then $\mu_1 = \mu_2$.

Proof. Apply Proposition 2.5 with $A_t = U_t$, $B_t = V_t$. ∎

Propositions 4.1 and 4.2 show that $\Omega_{-}\mu$ is the unique state asymptotic under V_t to the initial state $U_t\mu$.

Scattering Operator. The scattering operator S on $\mathscr{H}$ is defined as

$$S = \Omega_+^* \Omega_- \tag{38}$$

Under the assumptions mentioned above it is unitary. It therefore induces an automorphism of M according to

$$S\mu = \sum_{i \in I} \lambda_i \mu(Sf_i) \tag{39}$$

[cf. (30)].

Proposition 4.3. For any state μ,

$$\lim_{t \to +\infty} d(U_t S\mu, V_t \Omega_- \mu) = 0 \tag{40}$$

Proof. If $f \in \mathscr{H}$, and recalling that asymptotic completeness is assumed,

$$\|U_t Sf - V_t \Omega_- f\| = \|V_t^* U_t Sf - \Omega_- f\| = \|V_t^* U_t \Omega_+^* \Omega_- f - \Omega_- f\|$$

$$\underset{(t \to +\infty)}{\to} \|\Omega_+ \Omega_+^* \Omega_- f - \Omega_- f\| = \|\Omega_- f - \Omega_- f\| = 0$$

Since $A_t = U_t S$ and $B_t = V_t \Omega_-$ are isometric operators on $\mathscr{H}$ the result follows from proposition 3.6. ■

Suppose $U_t \mu_1$ and $U_t \mu_2$ are asymptotic to $V_t \Omega_- \mu$ when $t \to +\infty$. Then $d(V_t \Omega_- \mu, U_t \mu_1)$ and $d(V_t \Omega_- \mu, U_t \mu_2)$ both tend to zero when $t \to +\infty$. U_t is isometric, hence by Proposition 2.5 $\mu_1 = \mu_2$. Thus $U_t S\mu$ is the only state asymptotic to $V_t \Omega_- \mu$ as $t \to +\infty$, and so is *the* final state of free motion.

Proposition 4.4. If $p \in \mathscr{L}(\mathscr{H})$ then, for any μ in M,

$$\lim_{t \to +\infty} P[p|U_t S\mu] = \lim_{t \to +\infty} P[p|V_t \Omega_- \mu] \tag{41}$$

provided either side exists.

Proof.

$$|P[p|V_t S\mu] - P[p|V_t \Omega_- \mu]| \leqslant d(U_t S\mu, V_t \Omega_- \mu)$$

If $t \to +\infty$ the right-hand side tends to zero by (40), hence if either side of (41) exists so does the other, and they are equal. ■

5. POSITION AND MOMENTUM

In this section we consider the position and momentum of the particle when its motion is free. Let $\mathbf{r}$ be the position vector of the particle relative to a fixed origin O and $\mathbf{k}$ be its wave vector, so that its momentum is $\hbar\mathbf{k}$. If the particle is in a pure state $\mu(f)$ then $\mathbf{r}$ has a continuous distribution $|f(\cdot)|^2$ so that

$$P[\mathbf{r} \in \mathscr{R}|f] = \int_{\mathscr{R}} |f(\mathbf{r})|^2 \, d^3\mathbf{r} \tag{42}$$

If F is the Fourier transform on $\mathscr{H}$ $\mathbf{k}$ has a continuous distribution $|Ff(\cdot)|^2$ so that

$$P[\mathbf{k} \in \mathscr{R}|f] = \int_{\mathscr{R}} |Ff(\mathbf{k})|^2 \, d^3\mathbf{k} \tag{43}$$

Let $c_{\mathscr{R}}$ be the characteristic function of $\mathscr{R}$, and denote by $C_{\mathscr{R}}$ the operator of multiplication by $c_{\mathscr{R}}$. Thus

$$C_{\mathscr{R}} f(\mathbf{r}) = c_{\mathscr{R}}(\mathbf{r}) f(\mathbf{r}) \tag{44}$$

$C_{\mathscr{R}}$ is the projection operator which projects onto the subspace of functions of $\mathscr{L}_2(\mathbb{R}^3)$ which vanish almost everywhere outside $\mathscr{R}$. In terms of $C_{\mathscr{R}}$ equations (42) and (43) can be written

$$P[\mathbf{r} \in \mathscr{R}|f] = \|C_{\mathscr{R}} f\|^2 \tag{45}$$

$$P[\mathbf{k} \in \mathscr{R}|f] = \|C_{\mathscr{R}} Ff\|^2 = \|F^* C_{\mathscr{R}} Ff\|^2 \tag{46}$$

$C_{\mathscr{R}}$ is the projection operator corresponding to the proposition $\mathbf{r} \in \mathscr{R}$, and so we can write

$$E_{\mathbf{r} \in \mathscr{R}} = C_{\mathscr{R}} \tag{47}$$

$F^* C_{\mathscr{R}} F$ is also a projection operator. It projects onto the subspace $F^* C_{\mathscr{R}} F \mathscr{H} (\mathscr{H} = \mathscr{L}_2(\mathbb{R}^3))$ of functions in $\mathscr{L}_2(\mathbb{R}^3)$ whose Fourier transforms vanish almost everywhere outside $\mathscr{R}$. Thus

$$F_{\mathbf{k} \in \mathscr{R}} = F^* C_{\mathscr{R}} F \tag{48}$$

and so (45) and (46) are equivalent to

$$P[\mathbf{r} \in \mathscr{R}|f] = \|E_{\mathbf{r} \in \mathscr{R}} f\|^2, \qquad P[\mathbf{k} \in \mathscr{R}|f] = \|E_{\mathbf{k} \in \mathscr{R}} f\|^2$$

These results must, of course, be the case, since for a pure state defined by a unit vector f $P[p|f] = \|E_p f\|^2$.

Now **r** and **k** have continuous distributions in pure states and by Gleason's theorem every state μ has the form (3) with μ_i equal to a pure state $\mu(f_i)$ for each $i \in I$.

It follows from Proposition 2.6 that **r** and **k** have continuous distributions in every state μ. Further,

$$P[\mathbf{r} \in \mathscr{R}|\mu] = \sum_{i \in I} \lambda_i P[\mathbf{r} \in \mathscr{R}|\mu(f_i)] = \sum_{i \in I} \lambda_i \|C_{\mathscr{R}} f_i\|^2 \tag{49}$$

$$P[\mathbf{k} \in \mathscr{R}|\mu] = \sum_{i \in I} \lambda_i P[\mathbf{k} \in \mathscr{R}|\mu(f_i)] = \sum_{i \in I} \lambda_i \|F^* C_{\mathscr{R}} F f_i\|^2 \tag{50}$$

If the particle is moving freely

$$\begin{aligned} |C_{\mathscr{R}} F U_t f_i(\mathbf{k})|^2 &= \left| C_{\mathscr{R}}(\mathbf{k}) \exp\left(-\frac{i\hbar k^2 t}{2m} \right) F f_i(\mathbf{k}) \right|^2 \\ &= |C_{\mathscr{R}}(\mathbf{k}) F f_i(\mathbf{k})|^2 \\ &= |C_{\mathscr{R}} F f_i(\mathbf{k})|^2 \end{aligned}$$

so

$$\|C_{\mathscr{R}} F U_t f_i\|^2 = \|C_{\mathscr{R}} F f_i\|^2$$

Thus

$$\|F^* C_{\mathscr{R}} F U_t f_i\|^2 = \|F^* C_{\mathscr{R}} F f_i\|^2$$

and so from (50)

$$P[\mathbf{k} \in \mathscr{R}|\mu] = \sum_{i \in I} \lambda_i \|F^* C_{\mathscr{R}} F U_t f_i\|^2 = P[\mathbf{k} \in \mathscr{R}|U_t \mu] \tag{51}$$

Equation (51) shows that the probability distribution of momentum for a freely moving particle is independent of time, as is to be expected.

6. SCATTERING INTO CONES

Suppose the particle is moving freely. Let $\mathscr{C}$ denote a cone, vertex 0, and let $-\mathscr{C}$ denote the reflection of this cone in 0. It can be shown (Amrein, Jauch, and Sinha, 1977, p. 125) that if $f \in \mathscr{H}$ then

$$\lim_{t \to \pm\infty} P[\mathbf{r} \in \mathscr{C}|U_t f] = P[\mathbf{k} \in \pm\mathscr{C}|f] \tag{52}$$

That is,

$$\lim_{t\to\pm\infty} \|C_{\mathscr{C}}U_t f\|^2 = \|F^* C_{\pm\mathscr{C}} Ff\|^2 \tag{53}$$

Now

$$P[\mathbf{r}\in\mathscr{C}|U_t\mu] = \sum_{i\in I} \lambda_i \|C_{\mathscr{C}}U_t f_i\|^2 \tag{54}$$

$$P[\mathbf{k}\in\pm\mathscr{C}|\mu] = \sum_{i\in I} \lambda_i \|F^* C_{\pm\mathscr{C}} Ff_i\|^2 \tag{55}$$

Since the series (54) is uniformly convergent we can let $t\to+\infty$ term by term and then use (53) and (55) to obtain

$$\lim_{t\to\pm\infty} P[\mathbf{r}\in\mathscr{C}|U_t\mu] = P[\mathbf{k}\in\pm\mathscr{C}|\mu] \tag{56}$$

(56) shows that (52) is true for any state μ.

Suppose now that the interaction is present. The cone $\mathscr{C}$ is defined by a detector, and the probability that the particle enters the detector after the collision is therefore the probability of the particle ending up in $\mathscr{C}$ when $t\to+\infty$. This is

$$\lim_{t\to+\infty} P[\mathbf{r}\in\mathscr{C}|V_t\Omega_-\mu] \tag{57}$$

provided the limit exists. Now by Proposition 4.3 $d(U_tS\mu, V_t\Omega_-\mu)\to 0$ as $t\to+\infty$, and by (56)

$$\lim_{t\to+\infty} P[\mathbf{r}\in\mathscr{C}|U_tS\mu] = P[\mathbf{k}\in\mathscr{C}|S\mu]$$

Hence by Proposition 4.4 the limit (57) exists, and moreover

$$\lim_{t\to+\infty} P[\mathbf{r}\in\mathscr{C}|V_t\Omega_-\mu] = P[\mathbf{k}\in\mathscr{C}|S\mu] \tag{58}$$

(58) states that the probability of the particle being detected after the collision is the probability of the momentum being in the cone $\mathscr{C}$ in the final freely moving state $U_tS\mu$.

If (50) is applied to this case we find that

$$P[\mathbf{k}\in\mathscr{C}|S\mu] = \sum_{i\in I} \lambda_i \|C_{\mathscr{C}}FSf_i\|^2 \tag{59}$$

7. TRANSLATION OF STATES

Suppose in the first place that the motion of the particle is classical. The lattice $\mathscr{L}$ of propositions is in 1:1 correspondence with the Borel subsets of phase space, the space of 6-vectors $(\mathbf{r},\mathbf{p})$ where $\mathbf{r}$ is the position vector and $\mathbf{p}=\hbar\mathbf{k}$ is the momentum vector. If $\mathbf{r}$ varies over a Borel subset $\mathscr{R}$ of $\mathbb{R}^3$ and $\mathbf{p}$ varies over another Borel subset $\mathscr{S}$ of $\mathbb{R}^3$ we obtain a typical element $\mathscr{R}\oplus\mathscr{S}$ of $\mathscr{L}$, where $\mathscr{R}\oplus\mathscr{S}$ is the direct sum of $\mathscr{R}$ and $\mathscr{S}$.

If $T_{\mathbf{a}}\mathscr{R}$ is the translation of $\mathscr{R}$ through a vector displacement $\mathbf{a}$ we can define a space translation $T_{\mathbf{a}}$ of $\mathscr{L}$ in the classical case by

$$T_{\mathbf{a}}(\mathscr{R}\oplus\mathscr{S})=(T_{\mathbf{a}}\mathscr{R})\oplus\mathscr{S} \tag{60}$$

Since the proposition $\mathbf{r}\in\mathscr{R}$ corresponds to $\mathscr{R}\oplus\mathbb{R}^3$ while the proposition $\mathbf{p}\in\mathscr{S}$ corresponds to $\mathbb{R}^3\oplus\mathscr{S}$, we have

$$T_{\mathbf{a}}(\mathbf{r}\in\mathscr{R})\leftrightarrow T_{\mathbf{a}}(\mathscr{R}\oplus\mathbb{R}^3)=(T_{\mathbf{a}}\mathscr{R})\oplus\mathbb{R}^3\leftrightarrow \mathbf{r}\in T_{\mathbf{a}}\mathscr{R}$$

while

$$T_{\mathbf{a}}(\mathbf{p}\in\mathscr{S})\leftrightarrow T_{\mathbf{a}}(\mathbb{R}^3\oplus\mathscr{S})=(T_{\mathbf{a}}\mathbb{R}^3)\oplus\mathscr{S}=\mathbb{R}^3\oplus\mathscr{S}\leftrightarrow \mathbf{p}\in\mathscr{S}$$

Thus propositions concerning the possible values of the momentum are invariant under $T_{\mathbf{a}}$.

The automorphism $T_{\mathbf{a}}$ on $\mathscr{L}$ induces an automorphism $T_{\mathbf{a}}$ on $M(\mathscr{L})$. Since the inverse of $T_{\mathbf{a}}$ is $T_{-\mathbf{a}}$ we have, for all p in $\mathscr{L}$,

$$P[p|T_{\mathbf{a}}\mu]=P[T_{-\mathbf{a}}p|\mu] \tag{61}$$

(61) shows that, for any $\mathscr{R}\subseteq\mathbb{R}^3$,

$$P[\mathbf{r}\in\mathscr{R}|T_{\mathbf{a}}\mu]=P[\mathbf{r}\in T_{-\mathbf{a}}\mathscr{R}|\mu] \tag{62}$$

If $\mathbf{r}$ has a probability distribution $P[\cdot|\mu]$ (62) can be written

$$\int_{\mathscr{R}}P[\mathbf{r}|T_{\mathbf{a}}\mu]\,d^3\mathbf{r}=\int_{T_{-\mathbf{a}}\mathscr{R}}P[\mathbf{r}|\mu]\,d^3\mathbf{r} \tag{63}$$

Since $\mathbf{r}\in T_{-\mathbf{a}}\mathscr{R}$ if and only if $\boldsymbol{\rho}=\mathbf{r}+\mathbf{a}\in\mathscr{R}$ we have

$$\int_{T_{-\mathbf{a}}\mathscr{R}}P[\mathbf{r}|\mu]\,d^3\mathbf{r}=\int_{\mathscr{R}}P[\boldsymbol{\rho}-\mathbf{a}|\mu]\,d^3\boldsymbol{\rho}$$

and so (63) gives (since $\mathscr{R}$ is arbitrary)

$$P[\mathbf{r}|T_{\mathbf{a}}\mu]=P[\mathbf{r}-\mathbf{a}|\mu] \tag{64}$$

Let us now consider the quantum mechanical case. For any p in $\mathscr{L}(\mathscr{H})$,

$$P[p|\mu] = \sum_{i \in I} \lambda_i P[p|f_i] \tag{65}$$

The state $T_{\mathbf{a}}\mu$ must be defined by the expression, for all p in $\mathscr{L}$,

$$P[p|T_{\mathbf{a}}\mu] = P[T_{-\mathbf{a}}p|\mu] \tag{66}$$

If $p \Leftrightarrow \mathbf{r} \in \mathscr{R}$ then we naturally require $T_{-\mathbf{a}}p \Leftrightarrow \mathbf{r} \in T_{-\mathbf{a}}\mathscr{R}$ and so (65) and (66) yield

$$\begin{aligned} P[\mathbf{r} \in \mathscr{R}|T_{\mathbf{a}}\mu] &= P[T_{-\mathbf{a}}(\mathbf{r} \in \mathscr{R})|\mu] \\ &= P[\mathbf{r} \in T_{-\mathbf{a}}\mathscr{R}|\mu] \\ &= \sum_{i \in I} \lambda_i \int_{T_{-\mathbf{a}}\mathscr{R}} |f_i(\mathbf{r})|^2 \, d^3\mathbf{r} \end{aligned}$$

Since $\mathbf{r} \in T_{-\mathbf{a}}\mathscr{R} \Leftrightarrow \boldsymbol{\rho} = \mathbf{r} + \mathbf{a} \in \mathscr{R}$ we can take $\boldsymbol{\rho}$ as new variable of integration to obtain

$$P[\mathbf{r} \in \mathscr{R}|T_{\mathbf{a}}\mu] = \sum_{i \in I} \lambda_i \int_{\mathscr{R}} |f_i(\boldsymbol{\rho} - \mathbf{a})|^2 d^3\boldsymbol{\rho} \tag{67}$$

If we define $T_{\mathbf{a}}$ on $\mathscr{H}$ by, for any f in $\mathscr{H}$

$$T_{\mathbf{a}} f(\mathbf{r}) = f(\mathbf{r} - \mathbf{a}) \tag{68}$$

(67) becomes

$$P[\mathbf{r} \in \mathscr{R}|T_{\mathbf{a}}\mu] = \sum_{i \in I} \lambda_i \|C_{\mathscr{R}} T_{\mathbf{a}} f_i\|^2 \tag{69}$$

(69) can be taken as the definition of $T_{\mathbf{a}}\mu$—the only definition consistent with (66) and the interpretation of $T_{-\mathbf{a}}$ $(\mathbf{r} \in \mathscr{R})$ as $\mathbf{r} \in T_{-\mathbf{a}}\mathscr{R}$.

It follows that

$$\begin{aligned} P[\mathbf{k} \in \mathscr{R}|T_{\mathbf{a}}\mu] &= \sum_{i \in I} \lambda_i P[\mathbf{k} \in \mathscr{R}|T_{\mathbf{a}} f_i] \\ &= \sum_{i \in I} \lambda_i \|C_{\mathscr{R}} F T_{\mathbf{a}} f_i\|^2 \end{aligned} \tag{70}$$

Now it is easy to see from (68) that

$$FT_{\mathbf{a}}f_i(\mathbf{k}) = \exp(-i\mathbf{k}\cdot\mathbf{a})Ff_i(\mathbf{k}) \tag{71}$$

hence

$$\|C_{\mathscr{R}}FT_{\mathbf{a}}f_i\|^2 = \int_{\mathscr{R}} |FT_{\mathbf{a}}f_i(\mathbf{k})|^2\, d^3\mathbf{k} = \|C_{\mathscr{R}}Ff_i\|^2$$

so by (70)

$$\begin{aligned} P[\mathbf{k}\in\mathscr{R}|T_{\mathbf{a}}\mu] &= \sum_{i\in I} \lambda_i \|C_{\mathscr{R}}Ff_i\|^2 \\ &= P[\mathbf{k}\in\mathscr{R}|\mu] \end{aligned} \tag{72}$$

Thus the probability of the proposition $\mathbf{k}\in\mathscr{R}$ in the state $T_{\mathbf{a}}\mu$ is the same as its probability in the state μ, for every μ in M. We therefore conclude that $\mathbf{k}\in\mathscr{R}$ is invariant under $T_{\mathbf{a}}$.

8. THE INITIAL STATE

We suppose, as usual, that initially the particle is projected in the positive z direction. This enables us to assume that the initial state must satisfy certain conditions, which we now examine.

Provided the incident pulse is well collimated we can suppose that the momentum of the incident particle must lie inside a closed cone $\mathscr{C}_0$ of small acute semivertical angle α whose vertex is the origin and whose axis is the positive z axis. We can also assume that the incident pulse is very nearly monochromatic in energy. This means that we can assume that the magnitude $\hbar k$ of the momentum must lie between some positive number $\hbar k_1$ and some second, slightly larger, positive number $\hbar k_2$, so that

$$0 < k_2 - k_1 \ll k_1$$

Let $\mathscr{S}_{12}$ be the closed spherical shell

$$\mathscr{S}_{12} = \{\mathbf{k}\in\mathbb{R}^3 \colon k_1 \leqslant k \leqslant k_2\}$$

and suppose that $\mathscr{K}$ is a region of $\mathbb{R}^3$ lying entirely outside $\mathscr{S}_{12}\cap\mathscr{K}$. Then $P[\mathbf{k}\in\mathscr{K}|U_t\mu]$ must vanish for $t\to-\infty$, and so by (51) $P[\mathbf{k}\in\mathscr{K}|\mu]=0$. Hence by (51)

$$\sum_{i\in I} \lambda_i \|C_{\mathscr{R}}Ff_i\|^2 = 0$$

For each $i\in I$ $\lambda_i > 0$, hence $\|C_{\mathscr{K}}Ff_i\| = 0$ and so Ff_i vanishes almost everywhere in $\mathscr{K}$. It follows that Ff_i vanishes almost everywhere outside $\mathscr{S}_{12} \cap \mathscr{K}$ for each i in I.

9. CROSS SECTIONS

We have supposed that the incident particle is projected toward 0 in the positive z direction. Let us suppose further that this experiment is repeated N times per unit time, and the number of particles $N(\mathscr{C})$ emerging per unit time into a cone $\mathscr{C}$ of vertex 0 is counted—the flux into $\mathscr{C}$. For large N the flux $N(\mathscr{C})$ will be N times the probability that the particle emerges in $\mathscr{C}$. Since $\hbar\mathbf{k}$ is the momentum this probability is $P[\mathbf{k} \in \mathscr{C} | U_t S\mu] = P[\mathbf{k} \in \mathscr{C} | S\mu]$ where $U_t S\mu$ $(t \sim +\infty)$ is the final freely moving state. This is true also when the particle moves according to the laws of classical mechanics, since after the collision the particle trajectories are straight lines.

Suppose that the center of force producing the scattering is at 0, and let the center of force be translated through a vector displacement $\mathbf{a} = (a_x, a_y, 0)$, so that $\mathbf{a}$ is perpendicular to $0z$. This is equivalent to the translation of the incident beam (or pulse) through a vector displacement $-\mathbf{a}$, so that the state μ becomes $T_{-\mathbf{a}}\mu$. The initial (freely moving) state is now $U_t T_{-\mathbf{a}}\mu = T_{-\mathbf{a}} U_t \mu$ (since $T_{-\mathbf{a}}$ commutes with the *unperturbed* Hamiltonian). The flux into $\mathscr{C}$ after the collision now depends on $\mathbf{a}$, so let us denote it by $N(\mathscr{C}|\mathbf{a})$. Then

$$\begin{aligned} N(\mathscr{C}|\mathbf{a}) &= NP[\mathbf{k} \in \mathscr{C} | U_t S T_{-\mathbf{a}}\mu] \\ &= NP[\mathbf{k} \in \mathscr{C} | S T_{-\mathbf{a}}\mu] \end{aligned} \tag{73}$$

by (51).

Let $\mathscr{S}$ be a region of the plane $z = 0$, of area $A_{\mathscr{S}}$, and denote by $\overline{N}_{\mathscr{S}}(\mathscr{C})$ the average over $\mathscr{S}$ of $N(\mathscr{C}|\mathbf{a})$. Then

$$\overline{N}_{\mathscr{S}}(\mathscr{C}) = A_{\mathscr{S}}^{-1} \int_{\mathscr{S}} N(\mathscr{C}|\mathbf{a})\, d^2\mathbf{a} \tag{74}$$

From (73) and (74) we obtain

$$\overline{N}_{\mathscr{S}}(\mathscr{C}) = NA_{\mathscr{S}}^{-1} \int_{\mathscr{S}} P[\mathbf{k} \in \mathscr{C} | S T_{-\mathbf{a}}\mu]\, d^2\mathbf{a} \tag{75}$$

If $F_{\mathscr{S}}$ is the proportion of the incident pulse (or beam) before translation which passes through $\mathscr{S}$, $F_{\mathscr{S}} N A_{\mathscr{S}}^{-1}$ is the average incident flux across $\mathscr{S}$, which we shall denote by $\bar{I}_{\mathscr{S}}$. Dividing (75) by $\bar{I}_{\mathscr{S}}$ we obtain

$$\frac{\bar{N}_{\mathscr{S}}}{\bar{I}_{\mathscr{S}}} = F_{\mathscr{S}}^{-1} \int_{\mathscr{S}} P[\mathbf{k} \in \mathscr{C} | ST_{-\mathbf{a}}\mu] \, d^2\mathbf{a} \tag{76}$$

When $\mathscr{S}$ is allowed to expand to fill all of $\mathbb{R}^2$ the integral over a in $\mathscr{S}$ on the right-hand side of (76) cannot decrease. It therefore either tends to a limit or tends to infinity, while $F_{\mathscr{S}} \to 1$ from below. It follows that the left-hand side of (76) tends to a limit if, and only if, the integral over $\mathscr{S}$ tends to a limit. If this is the case we denote the limit [of either side of (76)] by $\sigma(\mathscr{C})$, and then

$$\sigma(\mathscr{C}) = \int P[\mathbf{k} \in \mathscr{C} | ST_{-\mathbf{a}}\mu] \, d^2\mathbf{a} \tag{77}$$

The quantity $\sigma(\mathscr{C})$ clearly has the dimensions of area.

It is interesting to examine the nature of $\sigma(\mathscr{C})$ in the special case when the incident beam is uniform and $\mathscr{S}$ is its cross section through 0. In practice we can usually assume that the range of the interaction is small compared with the dimensions of $\mathscr{S}$, so that edge effects can be neglected. It follows that the scattering is uniform over $\mathscr{S}$, and zero outside $\mathscr{S}$. Hence

$$P[\mathbf{k} \in \mathscr{C} | ST_{-\mathbf{a}}\mu] = \begin{cases} P[\mathbf{k} \in \mathscr{C} | S\mu] & (\mathbf{a} \in \mathscr{S}) \\ 0 & (\mathbf{a} \notin \mathscr{S}) \end{cases}$$

In this case, therefore, (77) yields

$$\sigma(\mathscr{C}) = \int_{\mathscr{S}} P[\mathbf{k} \in \mathscr{C} | ST_{-\mathbf{a}}\mu] \, d^2\mathbf{a}$$

Further $F_{\mathscr{S}} = 1$, and if $\mathbf{a} \in \mathscr{S}$, $N(\mathscr{C}|\mathbf{a}) = NP[\mathbf{k} \in \mathscr{C} | ST_{-\mathbf{a}}\mu]$ is independent of a, and equal to $N(\mathscr{C}) = NP[\mathbf{k} \in \mathscr{C} | S\mu]$, while $\bar{I}_{\mathscr{S}} = I$, the flux of the incident beam. Thus by (74) $\bar{N}_{\mathscr{S}}(\mathscr{C}) = N(\mathscr{C})$ so (76) implies that

$$\sigma(\mathscr{C}) = \frac{N(\mathscr{C})}{I} = \frac{\text{flux of particles into } \mathscr{C}}{\text{incident flux}} \tag{78}$$

If $\mathscr{C}$ is a cone of small semivertical angle, subtending a corresponding small angle $\Delta\omega$ at 0, (78) can be approximated by

$$\sigma(\mathscr{C}) \simeq \frac{d\sigma}{d\omega} \Delta\omega \tag{79}$$

where $d\sigma/d\omega$ is the differential cross section.

(79) relates $\sigma(\mathscr{C})$ to the differential cross section. It should be emphasized that both (78) and (79) represent approximations in ideal cases, and in what follows neither will be assumed to be necessarily valid. Nor, indeed, will the idealizations on which they are based—uniformity of the incident beam and negligible edge effects—be assumed. Our starting point will be (77).

10. CLASSICAL CALCULATION

In this section we shall examine the scattering of a steady beam of particles by a center of force if the particles obey the laws of classical mechanics. We shall use (77) to obtain an expression for $\sigma(\mathscr{C})$ in terms of the differential cross section.

Let the differential cross section for scattering of a particle of momentum $\hbar k\omega_0$ ($|\omega_0|=1$) into momentum $\hbar k\omega$ ($|\omega|=1$) be denoted by $d\sigma(k; \omega_0 \to \omega)/d\omega$. We shall suppose that the incident beam has a volume density of flux $I(k\omega_0, \mathbf{a})$ at the point $\mathbf{a}$ of $\mathscr{S}$. That is, if $(k\omega_0, k^2\,dk\,d^2\omega_0) = (\mathbf{k}_0, d^3\mathbf{k}_0)$ is a volume element of $\mathbf{k}_0$ space then the flux of particles with momentum in this volume element is

$$I(\mathbf{k}_0, \mathbf{a})\,d^3\mathbf{k}_0 = I(k\omega_0, \mathbf{a})\,k^2\,dk\,d^2\omega_0$$

in the direction of $\mathbf{k}_0$. Provided this flux varies by an negligible amount over the region of interaction this produces a flux of scattered particles into the solid angle $d^2\omega$ of amount

$$I(k\omega_0, \mathbf{a})\,k^2\,dk\,d^2\omega_0 \frac{d\sigma}{d\omega}(k; \omega_0 \to \omega)\,d^2\omega$$

hence the total flux into $\mathscr{C}$ is

$$\int_{\mathscr{C}} d^2\omega \int_0^\infty k^2\,dk \int_{\mathscr{C}_0} d^2\omega_0\, I(k\omega_0, \mathbf{a}) \frac{d\sigma}{d\omega}(k; \omega_0 \to \omega)$$

The probability $P[k\omega \in \mathscr{C} | ST_{-\mathbf{a}}\mu]$ of a particle being scattered into the cone $\mathscr{C}$ if the center of force is at $\mathbf{a}$ is this flux divided by the number of particles incident in unit time—that is, by N. Hence

$$P[k\omega \in \mathscr{C} | ST_{-\mathbf{a}}\mu] = N^{-1} \int_{\mathscr{C}} d^2\omega \int_0^\infty k^2\,dk \int_{\mathscr{C}_0} d^2\omega_0\, I(k\omega_0, \mathbf{a}) \frac{d\sigma}{d\omega}(k; \omega_0 \to \omega) \quad (80)$$

Now the number of particles with momentum in the volume element

$(k\boldsymbol{\omega}_0, k^2\,dk\,d^2\boldsymbol{\omega}_0)$ which cross an element $d^2\mathbf{a}$ of $\mathscr{S}$ in unit time is

$$I(k\boldsymbol{\omega}_0, \mathbf{a})k^2\,dk\,d^2\boldsymbol{\omega}_0\,d^2\mathbf{a}\cos\theta_0$$

where θ_0 is the angle between $\boldsymbol{\omega}_0$ and the positive z direction. Hence

$$k^2\,dk\,d^2\boldsymbol{\omega}_0\cos\theta_0\int I(k\boldsymbol{\omega}_0,\mathbf{a})\,d^2\mathbf{a}$$

= the number of incident particles with momentum in $(k\boldsymbol{\omega}_0, k^2\,dk\,d^2\boldsymbol{\omega}_0)$ which cross the plane $z = 0$ in unit time

= N times the probability of the momentum of the incident particle being in $(k\boldsymbol{\omega}_0, k^2\,dk\,d^2\,\boldsymbol{\omega}_0)$

$= NP[k\boldsymbol{\omega}_0|\mu]\,k^2\,dk\,d^2\boldsymbol{\omega}_0$

where $P[\cdot|\mu]$ is the probability density for $\mathbf{k}_0$ in the initial state $U_t\mu$. Therefore

$$\int I(k\boldsymbol{\omega}_0,\mathbf{a})\,d^2\mathbf{a} = N\sec\theta_0 P[k\boldsymbol{\omega}_0|\mu] \tag{81}$$

By (77) and (80)

$$\sigma(\mathscr{C}) = N^{-1}\int d^2\mathbf{a}\int_{\mathscr{C}} d^2\boldsymbol{\omega}\int_0^\infty k^2\,dk\int_{\mathscr{C}_0} d^2\boldsymbol{\omega}_0\,I(k\boldsymbol{\omega}_0,\mathbf{a})\frac{d\sigma}{d\omega}(k;\boldsymbol{\omega}_0\to\boldsymbol{\omega})$$

If we use Fubini's theorem to interchange the integrals and then use (81) to carry out the integration over $\mathbf{a}$ we obtain

$$\sigma(\mathscr{C}) = \int_{\mathscr{C}} d^2\boldsymbol{\omega}\left[\int_0^\infty k^2\,dk\int_{\mathscr{C}_0} d^2\boldsymbol{\omega}_0\,P[k\boldsymbol{\omega}_0|\mu]\,\frac{d\sigma(k;\boldsymbol{\omega}_0\to\boldsymbol{\omega})}{d\omega}\sec\theta_0\right] \tag{82}$$

The existence of the integrals, and the validity of Fubini's theorem, is assured if (i) $I(\cdot,\cdot)$ belongs to $\mathscr{L}_1(\mathbb{R}^5)$, (ii) $I(\cdot,\mathbf{a})$ has support in $\mathscr{S}_{12}\cap\mathscr{C}_0$ for almost all values of $\mathbf{a}$, (iii) the semivertical angle α of $\mathscr{C}_0$ is acute, (iv) $d\sigma(k;\boldsymbol{\omega}_0\to\boldsymbol{\omega})/d\omega$ is essentially bounded for $k\in[k_1,k_2]$, $\boldsymbol{\omega}_0\in\mathscr{C}_0$ and $\boldsymbol{\omega}\in\mathscr{C}$. The conditions on $\mathscr{C}_0$ and $d\sigma/d\omega$ are essential, for (82) shows that without either of them $\sigma(\mathscr{C})$ could be infinite.

11. QUANTUM CALCULATION

We shall now evaluate $\sigma(\mathscr{C})$ from (77) when the motion is quantum mechanical. Firstly we deal with the special case when μ is a pure state.

Then

$$P[\mathbf{k} \in \mathscr{C} | ST_{-\mathbf{a}}\mu] = P[\mathbf{k} \in \mathscr{C} | ST_{-\mathbf{a}} f] = \|C_{\mathscr{C}} FST_{-\mathbf{a}} f\|^2 \tag{83}$$

for some unit vector f in $\mathscr{L}_2(\mathbb{R}^3)$.

If $\mathscr{C} \cap \mathscr{C}_0 = \varnothing$ (so that no unscattered particles emerge in $\mathscr{C}$), $\|C_{\mathscr{C}} FT_{-\mathbf{a}} f\|^2 = P[\mathbf{k} \in \mathscr{C} | T_{-\mathbf{a}} f] = P[\mathbf{k} \in \mathscr{C} | f]$ [by (72)] $= 0$. Hence if $R = S - I$,

$$\begin{aligned} P[\mathbf{k} \in \mathscr{C} | ST_{-\mathbf{a}} f] &= \|C_{\mathscr{C}} FRT_{-\mathbf{a}} f\|^2 \\ &= \int_0^\infty k^2\, dk \int_{\mathscr{C}} d^2\omega |FRT_{-\mathbf{a}} f(k\omega)|^2 \end{aligned} \tag{84}$$

where $k\omega = \mathbf{k}$, $|\omega| = 1$.

We shall now *formally* evaluate $FRT_{-\mathbf{a}} f(k\omega)$ using the standard formulas of scattering theory for the S matrix [see, for example, equation (7.4.11) of Farina, 1973], and then *assume* the validity of the final result. We have

$$\begin{aligned} FRT_{-\mathbf{a}} f(k\omega) &= \langle k\omega | RT_{-\mathbf{a}} | f\rangle \\ &= \int_0^\infty k_0^2\, dk_0 \int d^2\omega_0 \langle k\omega | R | k_0\omega_0\rangle \langle k_0\omega_0 | T_{-\mathbf{a}} f\rangle \\ &= \int_0^\infty k_0^2\, dk_0 \int d^2\omega_0 (-2\pi i)\, \delta\left(\frac{\hbar^2 k^2}{2m} - \frac{\hbar^2 k_0^2}{2m}\right)\left(-\frac{\hbar^2}{4\pi^2 m}\right) \\ &\quad \times f(k; \omega_0 \to \omega) FT_{-\mathbf{a}} f(k\omega_0) \end{aligned}$$

where $f(k; \omega_0 \to \omega)$ is the scattering amplitude when the momentum of the particle changes from $\hbar k\omega_0$ to $\hbar k\omega$. That is,

$$FRT_{-\mathbf{a}} f(k\omega) = \frac{ik}{2\pi} \int d^2\omega_0 f(k; \omega_0 \to \omega) FT_{-\mathbf{a}} f(k\omega_0) \tag{85}$$

We shall assume that (85) holds rigorously, at least for values of k, ω, and ω_0, for which $f(k; \omega_0 \to \omega)$ is an $\mathscr{L}_2$ function of ω_0 and ω.

If we apply (71) to $T_{-\mathbf{a}} f$ we see that (85) becomes

$$FRT_{-\mathbf{a}} f(k\omega) = \frac{ik}{2\pi} \int d^2\omega_0 f(k; \omega_0 \to \omega) \exp(ik\omega_0 \cdot \mathbf{a}) Ff(k\omega_0) \tag{86}$$

Now we assume that $|f(k; \omega_0 \to \omega)|$ is bounded if $k \in [k_1, k_2]$, $\omega \in \mathscr{C}$, $\omega_0 \in \mathscr{C}_0$, so that it is certainly a square integrable function of ω over $\mathscr{C}$ and of ω_0 over $\mathscr{C}_0$. It follows that if $\omega \in \mathscr{C}$ then

$$f(k; \omega_0 \to \omega) Ff(k\omega_0)$$

is an $\mathscr{L}_2$ function of $k\omega_0$. We can therefore use Lemma 7.17 of Amrein, Jauch, and Sinha (1977, p. 284) to obtain

$$\int d^2\mathbf{a}\left|\frac{1}{2\pi}\int d^2\omega_0 \exp(ik\omega_0\cdot\mathbf{a})f(k;\omega_0\to\omega)\,Ff(k\omega_0)\right|^2$$

$$=\int d^2\omega_0\,|f(k;\omega_0\to\omega)|^2|Ff(k\omega_0)|^2k^{-2}\sec\theta_0$$

where, as in the last section, θ_0 is the angle between ω_0 and the positive z direction. It therefore follows from (86) that

$$\int d^2\mathbf{a}|FRT_{-\mathbf{a}}f(k\omega)|^2=\int_{\mathscr{C}_0} d^2\omega_0\,|f(k;\omega_0\to\omega)|^2|Ff(k\omega_0)|^2\sec\theta_0$$

since $Ff(k\omega_0)=0$ if $\omega_0\notin\mathscr{C}_0$. The right-hand side is integrable over ω in $\mathscr{C}$ and k in $[k_1,k_2]$, and vanishes if $k\notin[k_1,k_2]$, hence

$$\int_0^\infty k^2\,dk\int_{\mathscr{C}} d^2\omega\int d^2\mathbf{a}|FRT_{-\mathbf{a}}f(k\omega)|^2$$

$$=\int_0^\infty k^2\,dk\int_{\mathscr{C}} d^2\omega\int_{\mathscr{C}_0} d^2\omega_0\,|f(k;\omega_0\to\omega)|^2|Ff(k\omega_0)|^2\sec\theta_0$$

By Fubini's theorem the repeated integral on either side can be interchanged, so by (84) $\int d^2\mathbf{a}\,P[\mathbf{k}\in\mathscr{C}|ST_{-\mathbf{a}}\mu]$ exists, and moreover

$$\int d^2\mathbf{a}\,P[k\in\mathscr{C}|ST_{-\mathbf{a}}\mu]$$

$$=\int_{\mathscr{C}} d^2\omega\int_0^\infty k^2\,dk\int_{\mathscr{C}_0} d^2\omega_0\,|f(k;\omega_0\to\omega)|^2|Ff(k\omega_0)|^2\sec\theta_0 \tag{87}$$

By (77) the left-hand side of (87) is $\sigma(\mathscr{C})$, which therefore also exists. Further, since we have a pure state μ defined by f the probability density function in the initial state is given by

$$P[k\omega_0|\mu]=|Ff(k\omega_0)|^2$$

Hence (87) can be written

$$\sigma(\mathscr{C})=\int_{\mathscr{C}} d^2\omega\left[\int_0^\infty k^2\,dk\int_{\mathscr{C}_0} d^2\omega_0\,P[k\omega_0|\mu]|f(k;\omega_0\to\omega)|^2\sec\theta_0\right] \tag{88}$$

This is identical to the classical result (82) if we put

$$\frac{d\sigma}{d\omega}(k;\omega_0\to\omega)=|f(k;\omega_0\to\omega)|^2 \tag{89}$$

In fact (88) is true for any state. Firstly suppose that μ is given by the Gleason expansion (26) where I is finite. Provided $\mathscr{C}\cap\mathscr{C}_0=\varnothing$ we have $P[\mathbf{k}\in\mathscr{C}|T_{-\mathbf{a}}\mu]=P[\mathbf{k}\in\mathscr{C}|\mu]$ [by (72)] $=0$ whence $C_{\mathscr{C}}FT_{-\mathbf{a}}f_i=0$ for each i in I. Hence using (87) with μ replaced by μ_i

$$\begin{aligned}
&\int d^2\mathbf{a}\,P[\mathbf{k}\in\mathscr{C}|ST_{-\mathbf{a}}\mu]\\
&=\sum_{i\in I}\lambda_i\int d^2\mathbf{a}\,P[\mathbf{k}\in\mathscr{C}|ST_{-\mathbf{a}}\mu_i]\qquad[\text{by (26); } I \text{ is finite}]\\
&=\sum_{i\in I}\lambda_i\int_{\mathscr{C}}d^2\omega\int_0^\infty k^2\,dk\int_{\mathscr{C}_0}d^2\omega_0\,|f(k;\omega_0\to\omega)|^2|Ff_i(k\omega_0)|^2\sec\theta_0\\
&=\int_{\mathscr{C}}d^2\omega\int_0^\infty k^2\,dk\int_{\mathscr{C}_0}d^2\omega_0\,|f(k;\omega_0\to\omega)|^2\sec\theta_0\sum_{i\in I}\lambda_i|Ff_i(k\omega_0)|^2\\
&=\int_{\mathscr{C}}d^2\omega\int_0^\infty k^2\,dk\int_{\mathscr{C}_0}d^2\omega_0\,|f(k;\omega_0\to\omega)|^2\sec\theta_0 P[k\omega_0|\mu]
\end{aligned}$$

By (77) this is identical to (88).

We must now establish (88) when the index set I in (26) is infinite. Since $|f(k;\omega_0\to\omega)|^2$ is bounded for $k\in[k_1,k_2]$, $\omega\in\mathscr{C}$ and $\omega_0\in\mathscr{C}_0$, $\exists M>0$ such that, for these values, $|f(k;\omega_0\to\omega)|^2\leqslant M$. Then from (87), since $\sec\theta_0\geqslant\sec\alpha$ where α is the semivertical angle of $\mathscr{C}_0$, for each i and I,

$$\begin{aligned}
&\int d^2\mathbf{a}\,P[\mathbf{k}\in\mathscr{C}|ST_{-\mathbf{a}}\mu_i]\leqslant M\sec\alpha\int_{\mathscr{C}}d^2\omega\int_0^\infty k^2\,dk\int_{\mathscr{C}_0}d^2\omega_0\,|Ff_i(k\omega_0)|^2\\
&=M\sec\alpha\int_{\mathscr{C}}d^2\omega\int d^3\mathbf{k}_0\,|Ff_i(\mathbf{k}_0)|^2\\
&=M\sec\alpha\int_{\mathscr{C}}d^2\omega\qquad(\text{since }\|Ff_i\|=1)\\
&\leqslant 4\pi M\sec\alpha
\end{aligned}\tag{90}$$

Since μ is given by (26) with I infinite,

$$P[\mathbf{k} \in \mathscr{C} | ST_{-\mathbf{a}}\mu] = \sum_{i=1}^{\infty} \lambda_i p[\mathbf{k} \in \mathscr{C} | ST_{-\mathbf{a}}\mu_i] \tag{91}$$

Hence if $\mathscr{S}$ is a bounded region of the plane $z = 0$

$$\int_{\mathscr{S}} d^2\mathbf{a}\, P[\mathbf{k} \in \mathscr{C} | ST_{-\mathbf{a}}\mu] = \sum_{i=1}^{\infty} \lambda_i \int_{\mathscr{S}} d^2\mathbf{a}\, P[\mathbf{k} \in \mathscr{C} | ST_{-\mathbf{a}}\mu_i]$$

(since the series converges uniformly by comparison with $\sum_{i=1}^{\infty} \lambda_i = 1$)

$$\leqslant \sum_{i=1}^{\infty} \lambda_i \int d^2\mathbf{a}\, P[\mathbf{k} \in \mathscr{C} | ST_{-\mathbf{a}}\mu_i]$$

$$\leqslant \sum_{i=1}^{\infty} \lambda_i (4\pi M \sec\alpha) \qquad [\text{by } (90)]$$

$$= 4\pi M \sec\alpha$$

This shows that $\int P[\mathbf{k} \in \mathscr{C} | ST_{-\mathbf{a}}\mu]\, d^2\mathbf{a}$ exists, and hence by the dominated convergence theorem (91) may be integrated term by term to give

$$\begin{aligned} \int d^2\mathbf{a}\, P[\mathbf{k} \in \mathscr{C} | ST_{-\mathbf{a}}\mu] &= \sum_{i=1}^{\infty} \lambda_i \int d^2\mathbf{a}\, P[\mathbf{k} \in \mathscr{C} | ST_{-\mathbf{a}}\mu_i] \\ &= \sum_{i=1}^{\infty} \lambda_i \int_{\mathscr{C}} d^2\boldsymbol{\omega} \int_0^{\infty} k^2\, dk \int_{\mathscr{C}_0} d^2\boldsymbol{\omega}_0 \\ &\quad \times |f(k; \boldsymbol{\omega}_0 \to \boldsymbol{\omega})|^2 |Ff_i(k\boldsymbol{\omega}_0)|^2 \sec\theta_0 \end{aligned}$$

[by (87). Thus from (77)

$$\begin{aligned} \sigma(\mathscr{C}) = \sum_{i=1}^{\infty} \lambda_i \int_{\mathscr{C}} d^2\boldsymbol{\omega} \int_0^{\infty} k^2\, dk \int_{\mathscr{C}_0} d^2\boldsymbol{\omega}_0\, |f(k; \boldsymbol{\omega}_0 \to \boldsymbol{\omega})|^2 \\ \times |Ff_i(\mathbf{k}\boldsymbol{\omega}_0)|^2 \sec\theta_0 \end{aligned} \tag{92}$$

Now

$$P[k\boldsymbol{\omega}_0 | \mu] = \sum_{i=1}^{\infty} \lambda_i |Ff_i(k\boldsymbol{\omega}_0)|^2$$

may be integrated term by term over k in $[0, \infty)$ and $\omega_0 \in \mathscr{C}_0$, by the dominated convergence theorem. Since $|f(k; \omega_0 \to \omega)|^2 \sec\theta_0 \leqslant M \sec\alpha$ the same series can be multiplied by $|f(k; \omega_0 \to \omega)|^2 \sec\theta_0$ and integrated term over k in $[0, \infty)$ and ω_0 in $\mathscr{C}_0$ to yield

$$\int_0^\infty k^2\,dk \int_{\mathscr{C}_0} d^2\omega_0\,|f(k; \omega_0 \to \omega)|^2 P[k\omega_0|\mu] \sec\theta_0$$
$$= \sum_{i=1}^{\infty} \lambda_i \int_0^\infty k^2\,dk \int_{\mathscr{C}_0} d^2\omega_0\,|f(k; \omega_0 \to \omega)|^2 |Ff_i(k\omega_0)|^2 \sec\theta_0 \tag{93}$$

Since

$$\int_0^\infty k^2\,dk \int_{\mathscr{C}_0} d^2\omega_0\,|f(k; \omega_0 \to \omega)|^2 |Ff_i(k\omega_0)|^2 \sec\theta_0$$
$$\leqslant M \sec\alpha \int_0^\infty k^2\,dk \int_{\mathscr{C}_0} d^2\omega_0\,|Ff_i(k\omega_0)|^2$$
$$= M \sec\alpha \int d^3\mathbf{k}_0\,|Ff_i(\mathbf{k}_0)|^2$$
$$= M \sec\alpha$$

the series on the right-hand side of (93) is uniformly convergent, and so can be integrated term by term over $\omega \in \mathscr{C}$ to give

$$\int_{\mathscr{C}} d^2\omega \left[\int_0^\infty k^2\,dk \int_{\mathscr{C}_0} d^2\omega_0\,|f(k; \omega_0 \to \omega)|^2 P[k\omega_0|\mu] \sec\theta_0 \right]$$
$$= \sum_{i=1}^{\infty} \lambda_i \int_{\mathscr{C}} d^2\omega \int_0^\infty k^2\,dk \int_{\mathscr{C}_0} d^2\omega_0\,|f(k; \omega_0 \to \omega)|^2 |Ff_i(k\omega_0)|^2 \sec\theta_0$$

By (92) the right-hand side equals $\sigma(\mathscr{C})$, and so (88) is again obtained.

12. DISCUSSION

We have described a metric (4) on the convex set of states M on a complete orthocomplemented lattice. This enabled us to describe the asymptotic conditions of quantum scattering theory in the case of a particle moving under the influence of a center of force. We then noted properties that the initial state of a particle undergoing such a scattering process must have. The flux $\sigma(\mathscr{C})$ of particles per unit incident flux into a cone $\mathscr{C}$ after the scattering was related to an average of the probability of the final momentum being in $\mathscr{C}$, the average being over all positions of the center of force in a plane at right angles to the mean direction of incidence of the particle (77). This enabled us to relate $\sigma(\mathscr{C})$ to the differential cross section in the case of classical scattering (82), subject to the flux of the incident beam varying by a negligible amount over the interaction region. We then obtained, without this assumption, the expression (88) of $\sigma(\mathscr{C})$ in terms of the scattering amplitude when the particle obeys quantum mechanics. Equation (88) was obtained on the assumptions that (i) the initial momentum belongs to a cone $\mathscr{C}_0$ of acute semivertical angle α; (ii) the magnitude of the initial momentum is confined to a closed finite positive interval $[k_1, k_2]$ with zero as an exterior point; (iii) the scattering amplitude is bounded (or at least essentially bounded) for $k \in [k_1, k_2]$, $\omega \in \mathscr{C}$, $\omega_0 \in \mathscr{C}_0$; (iv) $\mathscr{C} \cap \mathscr{C}_0 = \varnothing$.

(iv) is necessarily true for a well-performed experiment, since without it we shall observe unscattered particles in the detector. (i) is also necessary, since without it the right-hand side of (88) could become infinite for certain choices of $P[\cdot|\mu]$; (iii) is also necessary for the same reason. (ii) is not really restrictive, since in an actual experiment the particles will usually, in practice, have positive lower and upper bounds on their energies.

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